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Abstract

The paper takes the Leibnitz Integral Rule (LIR) under variable integration limits and demonstrates how it can be applied to a business firm's dynamic problem of determining its optimum level of investment activity, when the longevity (life span) of the investment is itself a variable determined by the level of investment. Usually, the longevity of an investment is taken as given and fixed. Here, the level of investment activity has a dual effect, determining not only the longevity of the investment, but also the firm's net profit flow (which over the life span of the investment determines the firm's stock of wealth). The chosen application is the investment needed to create and maintain the salesperson-buyer business relationship involving the sale of a given product. The demonstration shows the assumptions needed to fit the LIR to the problem in order to make the fit tractable. The need for empirical research on the form of the longevity-investment function is also discussed.

Key Words: Leibnitz Integral Rule, Variable Integration Limits, Net Profit Flow, Level of Investment Activity, Dual Effect of Investment, and Longevity-Investment Function.

JEL Classification: C3, C31, H8, Z0, K160, D72

Introduction

The purpose of this paper is to demonstrate the application of the Leibnitz Integral Rule (hereafter, LIR) to longevity investment problems facing the business firm, when the limits of integration are variable and part of the optimizing process. The Rule essentially shows the equivalence between the derivative of an integral and the integral of a derivative (see, any textbook on advanced calculus). Typically, dynamic business investment decisions are based on fixed or given limits of integration. In our demonstration, the firm is faced with the task of deciding on what is the optimum longevity (or life span) for a given investment activity or task associated with selling a given product, when both the longevity (life span) and the yearly net profit flow are part of the optimizing process and subject to the same investment decision.

In other words, under variable integration limits, the decision variable has a dual effect on the accumulated wealth of the firm over the life span of the investment. On the one hand, an increase in the investment relationship (given by mL dollars, where L is the hours spent per year by the salesperson in establishing and maintaining the business relationship with the buyer and m is the cost or investment per L) lowers the net profit flow per year (say) and with all other things given, accumulated wealth is lowered, but, on the other hand, the longevity (or life span) of the net profit flow can increase ,so accumulated wealth is increased. For our demonstration, for simplicity we take the product quantity (given by Q) as fixed and focus on the investment task associated with selling the product and the longevity associated with the task. The LIR is appropriate for this dynamic situation for it uses variable limits on the integral, where the limits depend on the integrating decision variable (L). For our demonstration, the lower limit is taken as given (at the origin of the investment process, so the limit is fixed at zero), leaving the focus on the variable upper limit (or longevity). The details of the LIR and its application will be forthcoming.

We select, as an illustration of the use of the LIR, the business firm's task of determining and maintaining a seller-buyer relationship over time, in order to continue to sell the given product (for a simple example, consider a wholesale bread baking firm's salesperson(s) selling bread to a retail buying firm). The application task consists of renaming the variables in the canonical form of the LIR and making some simplifying assumptions about the form of the longevity function, T(L), to get at the essence of the application.

There are other economic and financial examples of the longevity issue of a relationship between two business entities, so while our choice is somewhat arbitrary, it has extensive literature that makes its use significant. The literature on this topic covers several disciplines including economics, marketing, psychology, and sociology (see, for a review of this extensive literature and eight types of seller-buyer relationships, Cannon and Perreault, 1999). Related and for a detailed account of the role of the key account salesperson see (Sengupta, Krapfel, and Pusateri, 2000). Also, for a review of the literature on the psychological trust in the relationship and its measurement see (Biong and Selnes, 1996; andYoung and Albaum, 2003). For an extensive review of the literature pertaining to the salesperson's characteristics, social aspects, value to buyer, and longevity of the salesperson-buyer relationship (see, Choi, Huang, and Sternquist, 2015). Our contribution to this literature is to show the LIR role in the dynamic process of determining the optimum longevity of the seller-buyer business relationship.

In what follows, in the next section, we summarize the general form of the LIR and what has to be changed to apply it. In the following section, we describe the wealth maximizing the model, outline its dynamic aspects, and show the optimizing solution process. The last section has a summary and conclusions.

2. *Summary of the general form of the LIR model and Changes Needed*

For an integral of the general form, F(x) = $\int_{a(x)}^{b(x)} f(x,t)dt$, the derivative of this integral is given by, d$\left(\int_{a(x)}^{b(x)} f(x,t)dt\right)$/dx = *f*(x, b(x))(db(x)/dx) - *f*(x, a(x))(da(x)/dx) +$\int_{a(x)}^{b(x)} (\frac{\partial f(x,t)}{\partial x})dt$. Since our origin is zero, the lower variable limit term drops out. It remains to show in the next section what changes are needed in the x and t variables for the application to fit the general form and what simplifying assumptions are useful to solve the duality relationship of variable L, as discussed earlier.

3. *Wealth Maximizing Model and Dynamic Aspects for an Optimum, L**

As indicated earlier, the key decision variable is the dual-effect L which affects both the net profit flow and the longevity, T. The salesperson's activity (consisting, for example, of time spent traveling, prep time, researching buyer's firm, dinners, movies, golf, gifts, and others) is indexed by "L" (in effect a vector) and investment cost to the firm is in "m" dollars per unit of L (as indicated earlier). The L in our model is measured in hours per time unit (years) of personal activity and contact (see, for an empirical study using regression methods on the time in hours spent in developing a personal social as opposed to business relationship between two persons, Hall, 2018). The seller-buyer business literature has empirical studies using Likert scale type data in factor analysis and studies using percentage of time spent selling (see, for a factor-analysis study of trust between seller and buyer, Young and Albaum, 2003, and, for a report on the percentage of time spent selling, see, Gschwandtner's review, 2011). But, these empirical results are difficult to use to specify the quantitative form of our longeity function, T(L). We find no literature on its exact empirical form, but Cannon and Perreault, Jr's 1999 study, Table 5, p. 450 shows mean averages of seller-buyer relationships of from 9.4 to 13.0 years.

Conceptually, the model is long run in perspective, where due to the expenditures on L, yearly net profit is less, but, on the other hand, due to L, longevity T(L) is longer, so the aggregate net profit (or wealth) over the time-period 0 to T(L) can be larger. This perspective is analogous to the stock-flow concept of many economic and business models, where here the aggregate net profit or wealth is the stock and net profit is the flow. Of equal importance is the buyer's side of the relationship, but to contain our model, we take the buyer's role as passive and given.

As in traditional economics and finance, the firm's yearly gross profit function is given by

(1) $p(Q) = R(Q) – C(Q),$

where "p" is gross profit for the given output, Q, R is revenue determined by the given price of the product, and C is the production cost (the time variable, t, is suppressed). The gross profit is given and fixed for the application given here. The investment cost of creating and maintaining the business relationship is based on the contact hours of the salesperson-buyer relationship L which determines the firm's net profit given by

(2) $\acute{p}(L, Q) = p(Q) - mL$,

where $\acute{p}(L,Q)$ is net profit function after accounting for L and m is the given price of L to include all personal relationship expenditures. The aggregate of (2) is taken over the time-period 0 to T(L) and given by

(3) $W(T(L), Q) = \acute{p}(L, Q)\cdot T(L)$,

where the function $\acute{p}(L, Q) = \acute{p}$ and W(.) is the long-run wealth created by the L effect on T and ṕ. From (3), Q in effect occurs yearly over a period of T years as does L, so treating the net profit function as a homogeneous function, then aggregate net profit is $W = \acute{p}\cdot T(L)$, aggregate output is $Q\cdot T(L)$, and aggregate contact hours are $L\cdot T(L)$.

In the dynamic model, the firm's goal is to choose a yearly contact level for L (which affects both the yearly net profit stream and the longevity of the seller-buyer business relationship) such that the discounted present value of wealth (aggregated net profit) over the longevity period is maximized. Still keeping Q as given and ignoring the discount factor (e^{-rt}) for simplicity, the integral form of the firm's objective function after relabeling the LIR variables for the given application is given by

(5) $W(T(L), Q) = \int_0^{T(L)} f\big(L,\ T(L)\big)dT(L)$,

where the upper time limit is given by T(L), the longevity function. The function *f(L, T(L))* is now labeled the marginal wealth (net profit) and equals ṕ(L)T(L). The t and dt were changed to T(L) and dT(L) as indicated earlier. The value of the firm's wealth over the time-period 0 to T(L) is, in effect, the aggregate of its marginal wealth over that time period.

In the dynamic equation (5), we are looking for the optimum length of the contact hours given by L*, for each year of the longevity given by T(L*). While each yearly seller-buyer contact between the salesperson and the buyer could have a different length in terms of contact hours, to simplify the structure of the model, we assume that L* is optimum in length with the same given number of contact hours per year for each year of the longevity period of T years. That is to say, optimum L* remains constant over the business contract period and is necessary to maintain the existence of the business contract's life.

The optimum L*, in general terms (following the Leibnitz Rule), is given by the first-order condition

(6) dW(T(L), Q)/dL = $\frac{d}{dL}\int_0^{T(L)} f(\mathrm{L},\mathrm{T(L)})\,d\mathrm{T(L)}$ = ṕ(L).T(L))(dT(L)/dL) +

$$\int_0^{T(L)} \left\{\frac{\partial f(\mathrm{L},\mathrm{T(L)})}{\partial L}\right\} dT(L)\,/dL = 0,$$

where the longevity function T(L) and its derivatives, the dT(L)/dL, and the partial $\frac{\partial f(\mathrm{L},\mathrm{T(L)})}{\partial \mathrm{L}}$,all require empirical identity for (6) to be solved numerically. We can pursue an algebraic solution by first recognizing that *f*(L, T(L)) = (π – mL)T(L) in the first term of (6). The second term, the partial, is more difficult to handle.

With the re-labeled form, the right-side LIR partial derivative in (6) becomes $\partial f(L, T(L))/\partial L$ = ∂[(p - mL)T(L)]/∂L, which equals (p -mL)(∂T(L)/∂L - mT(L). Simplifying further, let T(L) = sL and then let s =1, so T=L. The partial now becomes (p - 2mL). Integration wrt L gives $(pL - mL^2)$. The new first-order condition is now

(6') $(p - mL)L + (pL - mL^2) = 0,$

$2(pL - mL^2) = 0,$

$p – ml = 0,$

so the optimum longevity L* is given by the formula, L* = p/m.

The various simplifications turned the dual effect of L into a single effect with a solution. The resultant formula may be intuitive. Once the quantitative form of the longevity function T(L) is hypothesized, the T*(L*) can be estimated. As effort is made to increase the gross profit flow, the life of the business relationship can increase and the firm's wealth will increase. Any increase in the cost of the investment, m, will have the opposite effect on the longevity of the salesperson-buyer business relationship.

4. *Summary and Conclusions*

The Leibnitz Integration Rule under variable integration limits was fitted to the salesperson-buyer business relationship to determine the optimum longevity or life of the business relationship. With a number of simplifying assumptions, the dual effect of the salesperson's contact hours with the buyer was reduced to a manageable single effect, which allowed for an optimum wealth-maximizing longevity.

Taking the liberty of making simplifying assumptions to get at the essence of the problem is not unusual in mathematics or in economics and finance. In any case, the problem for future research is determining the quantitative form of the longevity function.

The importance of the form of the longevity function, T(L), to the validity and usefulness of equations (5), (6), and (6’) should be clear. It is possible that T(L) will have an upper limit, regardless of L contact hours. The L itself could have an upper limit (physically and psychologically), as implied by the psychological literature (See, Hall, 2018). In any case, empirical work is needed to shed light on the empirical form of the longevity function. In general, the application of the Leibnitz Rule should prove useful for investment problems when the optimum longevity of the investment is a variable to be determined by the optimization process.

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