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Nash's Bargaining Formula Revisited

Abstract: The note consists of showing the marginal rates of substitution (MRS) explicitly in Nash's product formula and contract curve given by the Jacobian determinant, which otherwise are contained therein only implicitly. Two simple examples from duopoly and bilateral monopoly are used to show the MRS's explicitly in the mathematical structure of the bargaining game. We learn from the note that, while the MRS's are implicit in the mathematical structure of the game, the MRS's after suitable transformation can be shown explicitly in the product formula and contract curve. This MRS showing puts the structure of the game in a context familiar to economists.

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1 Introduction

In a now famous paper, John Nash [1] demonstrated that given the assumptions of invariance of utility functions, Pareto Optimum, symmetry, and independence of irrelevant alternatives, the equilibrium solution to a bargaining problem will be given by the maximization of the product of the utilities of the two players in a cooperative game. And, further, the product formula is the only formula consistent with these assumptions (see [2], for this rendition).

In this brief note, the focus is on the Pareto Optimum condition as defined by the equality of the marginal rates of substitution (MRS) of the players (see, [3] and [4]). I demonstrate using as an example the traditional and simple Cournot–Nash duopoly bargaining problem in the output space that the product formula, namely, the product of the profit functions (utility functions), under maximization does indeed implicitly contain the Pareto Optimum condition as defined by the equality of the MRS's of the two firms. In other words, I make the MRS's appear explicitly in the mathematical structure of the bargaining game. The mathematical structure is transformed into a, in effect, verbal structure familiar to economists. Indirectly, the example also shows why/how the product formula works. This note is not a proof of the Nash assertion. Nash and others have done this amply. It simply shows how the mathematical structure can be transformed.

It is hoped that the note may influence the way Nash bargaining is presented in both upper division and graduate-level courses in microeconomics.

2 Pareto Optimum condition and the contract curve

In the context of the simple duopoly game, I show below that the equality of the MRS's is given by the vanishing determinant of the Jacobian matrix for the two profit functions of the two firms.

It is convenient to show the Jacobian form of the Pareto Optimum condition first. In general terms, the determinant of the Jacobian for the two profit functions is given by

$$|J| = \begin{vmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = (ad - bc) = 0, \tag{1}$$

indicating that the two profit functions are dependent. The profit for each firm, $p_1(q_1, q_2)$ and , $p_2(q_1, q_2)$, depends on its output and its rival's output (q_1 and q_2). The right-hand form of the determinant gives the Edgeworth contract curve (CC). The CC is the set of all output points (q_1, q_2), where the firms' MRS's are equal. Upon rearranging the right-hand form, the equality of the MRS's is obtained and given by

$$a/b = c/d = [\mathrm{d}q_2/\,\mathrm{d}q_1]_i = \mathrm{MRS}_i(i = 1, 2). \tag{2}$$

To show (2) explicitly for the simple example, let the market demand be $P = 1 - q_1 - q_2$, costs be zero, $\Pi_1 = (1 - q_1 - q_2)q_1$, and, $\Pi_2 = (1 - q_1 - q_2)q_2$. Then,[1]

$$\begin{aligned} |J| &= (1 - 2q_1 - q_2)(1 - q_1 - 2q_2) - q_1q_2 = 0 \\ &= (q_1M_1)(q_2\mathsf{M}_2) - q_1q_2 = 0 \\ &= M_1\mathsf{M}_2 - 1 = 0, \end{aligned} \tag{3}$$

where M_2 is the inverse of MRS_2. Thus, $M_1/M_2 = 1$, and the MRS's are equal all along the CC (a well understood result) and shown explicitly for the Jacobian structure. My point here is simply to show explicitly that the Jacobian determinant contains the Pareto Optimum condition. It is not intuitively obvious that this is indeed the case.

3 Nash product formula and the pareto optimum condition

Next, I demonstrate for the simple example that the product formula [3] implicitly contains the two MRS's. Let the product of the two profit functions be given by

$$\begin{aligned} W(q_1, q_2) = \Pi_1\Pi_2 &= [(1 - q_1 - q_2)q_1][(1 - q_1 - q_2)q_2] \\ &= (1 - q_1 - q_2)(1 - q_1 - q_2)q_1q_2. \end{aligned} \tag{4}$$

Insert dummy terms, $(2q_1 - 2q_1)$ and $(2q_2 - 2q_2)$ into the first and second products respectively to obtain

$$\begin{aligned} W &= [(1 - 2q_1 - q_2 + q_1)(1 - 2q_2 - q_1 + q_2)]q_1q_2 \\ &= [q_1(M_1 + 1)q_2(\mathsf{M}_2 + 1)]q_1q_2 \\ &= [(M_1 + 1)(\mathsf{M}_2 + 1)](q_1q_2)^2, \end{aligned} \tag{5}$$

where as before M_2 is the inverse of the MRS_2. Thus, the product formula contains the MRS's implicitly, but (5) now shows them explicitly. However, they are not necessarily equal. They are only equal at the equilibrium point on the CC where the iso-W(.) curve in the q-space is tangent to the CC, under constrained maximization.

To complete the demonstration, as is also well-known, the cartel solution for the simple example under joint-profit maximization is $Q = q_1 + q_2 = 1/2$ and $P = 1/2$. Joint profit is given by $PQ = 1/4 = \Pi_1 + \Pi_2$. The bargaining problem is to determine a "fair" (in Nash's sense) distribution of the maximum profit. Formally, the maximization or optimization problem can be set up in Lagrangean form and given by[2]

$$L(q_1, q_2, \lambda) = \Pi_1(\,.\,)\Pi_2(\,.\,) + \lambda(1/2 - q_1 - -q_2). \tag{6}$$

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1 The total differential for firm 1's profit function $\Pi_1 = (1-q_1-q_2)q_1$ is $\mathrm{d}Pi_1 = \mathrm{d}q_1 - 2q_1\,\mathrm{d}q_1 - q_2\,\mathrm{d}q_1 - q_1\,\mathrm{d}q_2 = 0$. Upon rearrangement, $\mathrm{d}q_2/\,\mathrm{d}q_1 = (1 - 2q_1 - q_2)/q_1 = M_1$, the MRS for firm 1. In a similar way, $\mathrm{d}q_2/\,\mathrm{d}q_1 = q_2/(1 - 2q_2 - q_1) = M_2$, the MRS for firm 2. Note, however, that in the derivations in the text, the inverse of M_2 (namely, M_2) has to be used first to be consistent with the use of the dummy terms.

2 The Jacobian determinant given by (2) for the simple example is $(1 - 3q_1 - 3q_2 + 4q_1q_2 + 2(q_1)^2 + 2(q_2)^2) = 0$. In spite of the non-linear appearance of this CC, its total differential shows it has a constant slope of –1. Thus, the constraint on (6) is given simply as linearly.

The maximization of (6) gives from the first-order conditions the equilibrium solution in the q-space. The first-order conditions after eliminating λ are set equal and given by

$$\begin{aligned}(1-2q_1-q_2)(q_2-q_2^2-q_1q_2)+(q_1-q_1^2-q_1q_2)(-q_2) &= \\ (1-2q_2-q_1)(q_1-q_1^2-q_1q_2)+(q_2-q_2^2-q_1q_2)(-q_1).\end{aligned} \tag{7}$$

Again, by inserting the previously used dummy terms into the appropriate profit functions and noting the definitions of the MRS's, (7) after somewhat tedious rearrangements can be shown to be

$$\begin{aligned} q_2(M_1+1)(\mathrm{M}_2+1)-q_1(M_1+1)(\mathrm{M}_2+1) &= 0, \\ q_2-q_1 &= 0. \end{aligned} \tag{8}$$

With $q_2 = q_1$ then from the constraint, both equal $1/4$. The optimum distribution of the joint profit is thus $Pq_i = 1/8$ to each firm. It can also be shown with another example that the equal MRS' are also equal to one in equilibrium for the simple data used.[3]

Thus, the left side of (6) shows the product formula that implicitly contains the MRS's as shown and the right side of (6) insures that the solution is on the CC, given by the Jacobian.

The interesting point about this whole optimization process is that while the MRS's are equal and embedded in the CC and are also embedded in the product formula, in equilibrium they are equal.

4 Summary and conclusion

To summarize, the Jocobian has equal MRS's and the product formula implicitly contains the MRS's. At the tangency (equilibrium in the output space), the MRS's in the product formula are equal since they have a common point on the CC. In the simple example the MRS's are also equal to one. The Nash assertion is that the product formula satisfies the Pareto Optimum condition. How is this so? It implicitly contains the MRS's and satisfies the condition in equilibrium. The note shows the MRS's explicitly.

What can be learned from the above demonstration? First and foremost, the use of the Jacobian determinant brings out explicitly the MRS's after proper transformation and contains the contract curve. Also, by a similar transformation, the product formula was shown to contain the MRS's. The constrained optimization links these two sets of MRS's together. Thus, we obtain a unified treatment of the two elements of the bargaining game by virtue of the appearance of the MRS's explicitly.

References

[1] Nash JF. The bargaining problem. Econometrica 1950,18,155–162.
[2] Luce RD, Raiffa H. Games and decisions: introduction and critical survey. New York, John Wiley & Sons, Inc., 1957.
[3] Nash JF. Two-person cooperative games. Econometrica 1953,21,128–141.
[4] Mayberry JP, Nash JF, Shubik M. A comparison of treatments of a duopoly situation. Econometrica 1953,21,141–154.
[5] Friedman JW. Game theory with applications to economics. New York, Oxford University Press, 1986.

3 A similar demonstration is possible using the firm-union bilateral monopoly example in [5, pp. 179–180]. With firm profit $= L(100 - L) - wL$ and union utility $= \sqrt{Lw}$, where L is employment and w is the wage rate, the vanishing Jacobian determinant in the $L - w$ space after rearrangements like in the text is $(\mathrm{MRS}_f - \mathrm{MRS}_u) = 0$. The $\mathrm{MRS}_f = (100 - 2L - w)/L = \mathrm{d}w/\mathrm{d}L$ and the $\mathrm{MRS}_u = -w/L = \mathrm{d}w/\mathrm{d}L$. The product of the two functions in the $L{-}w$ space is $L^3(\mathrm{MRS}_f+1)(i)\sqrt{\mathrm{MRS}_u}$, where $(i) = (\sqrt{-1})$. In equilibrium, the two MRS's in the product formula are equal so it ultimately is given by $L^3(1-w/L)\sqrt{w/L} = L(L-w)\sqrt{Lw} = 48,117$, Friedman's results. Thus, the MRS's implicitly in the product formula map into the equal MRS's implicitly in the Pareto Optimum function.