There are 100 points possible on this exam, 50 points each for Prof. Lozada's questions and Prof. Dugar's questions. Each professor asks you to do two long questions (worth 20 points each) and one short question (worth 10 points each). Each professor gives you a choice of which of his short questions to do.

There are three sections on this exam:

- In the first section contains all of the required questions. There are four of them. The first two are worth 20 points each; the second two are worth 19 points each.
- In the second section there are two questions; you should work one of them. Each is worth 12 points.
- In the third section there are two questions; you should work one of them. Each is worth 10 points.

You have 4 hours and 30 minutes (that is, until 1:30 PM) to finish this test. This gives you about 45 minutes per question.

Do not use different colors in your answers because we grade looking at black-and-white photocopies of your exam.

It would be helpful for you to put the number of the problem you are working on at the top of every page of your answers, so we do not accidentally ignore part of your answer.

In this document, v and w respectively denote the Roman lower-case "v" and "w." Also, in this document some questions begin on one page and end on the next page; therefore, do not assume that a question ends at the bottom of a page, but check to determine whether it continues onto the next page.

Good luck.

Section 1.
Answer all of the following four questions.

1. **[20 points]** Consider a perfectly competitive firm which purchases labor l and machines m, and with these inputs produces an output in the amount of $\tilde{\chi} \cdot l \cdot m$, where $\tilde{\chi}$ is a random variable representing uncertain knowledge of the production function. (The notation is inspired by the fact that "χ" is pronounced in a similar way to the third and fourth letters in "technology.") Output sells for a price p. Labor costs w per unit and the price of machines is p_m per unit.

   Unlike the firms discussed in the textbook, this firm is constrained in how much it can spend on labor and on machines. It can only spend at most CR dollars on labor and on machines, where CR, which is exogenous to the firm, is the amount of credit extended to the firm by its lenders. Its total net costs are the amount it spends on hiring labor, the amount it spends on buying machines, and $r \cdot CR$, where $r > 0$ is the interest rate and hence $r \cdot CR$ is the amount of interest which the firm has to pay back to its lenders. (The loan principal, CR, is both part of total revenue and part of total cost, so it cancels out of the expression for profit.)

   (a) State mathematically the firm's optimization problem.

   (b) Suppose the owner of the firm believes $\tilde{\chi}$ will take the value one with probability one. Under this assumption, find the optimal amount of labor l^* and machines m^* which the owner will hire. You need not check second-order conditions. Assume the spending constraint is satisfied with equality.

   (c) Prove that the optimized level of profit, π^*, given the firm owner's expectation that $\tilde{\chi} \equiv 1$, is

   $$\pi^* = \left[\frac{p\,CR}{4 p_m w} - (1 + r) \right] CR\,.$$

   (d) Suppose the owner's opportunity cost of running this firm is denoted by "OOC." What inequality must be satisfied in order for the owner to decide to operate this firm?

   (e) If $OOC = 0$, what values of CR are necessary to induce the owner to operate this firm?

(f) Assuming that the owner rationally decides that he should operate the firm (rather than engaging in an alternative activity), prove that π^* is increasing in CR. Explain in non-mathematical terms what this implies for the demand for credit by the owner of the firm.

(g) Suppose the supplier of credit to the firm is a bank. The bank creates credit "out of thin air" (out of nothing); it incurs no cost in creating credit. (This is because the loan expands both the asset and the liability side of the bank's balance sheet, but you do not need to know that.) The bank is not as optimistic as the firm owner about the value of $\tilde{\chi}$. Instead, the bank believes $\tilde{\chi}$ will be zero with probability $pr_b > 0$ ("bad probability") and will be one with probability $1 - pr_b < 1$. Assume that whether $\tilde{\chi}$ is 0 or 1 will only be known after the bank lends CR to the firm and after the firm has used CR to purchase l and m.

Argue that in "the bad state of the world" (that is, when $\tilde{\chi}$ turns out to be zero), the firm will be bankrupt and the bank will lose CR dollars.

(h) Suppose that the bank maximizes

$$pr_b \ln[\mathscr{W}_0 - CR] + (1 - pr_b) \ln[\mathscr{W}_0 + r\, CR]$$

where $\mathscr{W}_0 > 0$ is the bank's initial amount of wealth. (This comes from assuming the bank is risk-averse and maximizes its expected utility, but you do not need to know that.)

i. Argue from the bank's objective function that the bank's optimal level of CR, denoted by CR^*, satisfies $CR^* < \mathscr{W}_0$.

ii. Prove that

$$CR^* = \left(1 - pr_b - \frac{pr_b}{r}\right) \mathscr{W}_0 \,.$$

You do not have to check second-order conditions.

iii. Considering parts (h)(ii) and (f), would the firm owner prefer to borrow from a bank with large $\mathscr{W}_0$ or from a bank with small $\mathscr{W}_0$?

iv. Prove that $CR^* > 0$ and $r > 0$ imply

$$pr_b < \frac{r}{1+r} < r \,.$$

(i) Show that if the owner rationally decides that he should operate the firm (rather than engaging in an alternative activity), and if $OOC = 0$, then

$$\left(1 - pr_b - \frac{pr_b}{r}\right) \mathscr{W}_0 > (1 + r)\,\frac{4wp_m}{p}\,.$$

(j) The "expected value" (not "expected utility") of the bank is

$$EV_{bk} = pr_b\,(\mathscr{W}_0 - CR^*) + (1-pr_b)\,(\mathscr{W}_0 + r\,CR^*)\,.$$

i. Make a conjecture about the sign of $\partial EV_{bk}/\partial pr_b$. As always, explain your answer.

ii. [If you are running short on time, I recommend skipping this part because it takes a while and it's not worth very much.] Prove that the conjecture you just made is correct. Hint 1: it is easier to work with $\partial(EV_{bk}/\mathscr{W}_0)/\partial pr_b$. Hint 2: use (h)(iv).

(k) Briefly argue *against* the following opinion: "in this model, it is easy to define 'the rate of profit,' and 'the rate of profit' is equal to 'the return on lending money'." (There is more than one correct way of answering this question.)

2. **[20 points]** Consider a two-person two-commodity competitive general equilibrium in which the two people are denoted by 1 and 2, the two goods by x and y, and person i consumes x_i of good x and y_i of good y. Suppose the utility functions of the two individuals are

$$U_1 = x_1^{\alpha}\,y_1^{\beta} \quad \text{and}$$
$$U_2 = x_2^{\gamma}\,y_2^{\delta}\,.$$

Suppose α, β, γ, and δ are all positive. Suppose both people have 1 unit of good x and 1 unit of good y as their endowment (in other words, $\omega_1 = \omega_2 = (1, 1)$).

What effect will an increase in α have on the amount of good y which person 2 consumes?

Hint: The following answer is *wrong:* "an increase in α has no effect on the amount of good y which person 2 consumes."

3. **[19 points]** Consider the following signaling game. An item is of high quality with probability 1/3 and of low quality with probability 2/3. The seller of the item is privately informed of its quality.

The seller moves first, choosing whether to advertise ("adv") or not ("none"). The buyer observes whether or not the seller advertises and then chooses whether or not to buy. The price is fixed at \$3, whether the seller advertises or not. The table below shows how the cost of advertising and the buyer's value for the item depends on its quality.

	Quality	
	High	Low
Buyer Value	\$4	\$2
Cost of Advertising	\$1	\$4

The seller's payoff is sales revenue minus advertising costs (if any). The buyer's payoff is the difference between his value for the item and the price he pays.

(a) Sketch the extensive form representation of the game with payoffs for both the players. (8 points)

(b) Find all the pure strategy Perfect Bayesian equilibria (PBE) of the game. Show all your work for all possible pure strategy profiles whether or not that profile is an equilibrium. (11 points)

4. **[19 points]** Apu has hired a mechanic, called Dipu, to do a job using Apu's raw materials. However, Dipu can, by expending effort, reduce the amount of materials used. There are two types of mechanics, labeled $i = 1, 2$. Type i has private information about his efficiency parameter f_i, and $f_1 < f_2$. The probability that Dipu is type i is p_i, where each p_i is positive and $p_1 + p_2 = 1$. If type i expends effort e_i, the cost of raw materials used will be $c_i = f_i - e_i$. Dipu's effort is not verifiable. The fact that the job is completed and the cost of materials used are observable and verifiable. The type f_i is Dipu's private information, so Apu cannot infer the effort e_i from observation of the cost c_i. The utility function of Dipu regardless of his type is $u = t - e$, where t is the payment Dipu receives from Apu and e is the effort Dipu expends. The outside opportunity of the mechanic of either type is 0.

(a) What kind of information asymmetry is involved in this situation? Moral hazard, or adverse selection, or both? (1 point)

Apu offers a menu of two contracts, where he intends type i to select contract i.

(b) What is the form of the contracts, compatible with the information limitations? (2 points)

(c) What are the incentive compatibility constraints on Apu's contract choices? (4 points.)

(d) What are the participation constraints on Apu's contract choices? (4 points).

Apu wants to minimize his expected expense consisting of the payment to Dipu and the cost of materials,

$$p_1\,(t_1 + c_1) + p_2\,(t_2 + c_2)\,.$$

(e) Among the contracts satisfying the incentive compatibility and participation constraints for both types, show that Apu's expected expense is the lowest when $t_1 + c_1 = t_2 + c_2 = f_2$. (4 points).

(f) Contrast this with the hypothetical first-best case where Apu knows the type of Dipu. Compare Apu's expected cost and the two types' utilities in the two cases. (4 points).

Section 2.
Answer one of the following two questions.

1. **[12 points]** Apu is an expected utility maximizer who has a concave utility function $u(W)$ defined over his final wealth. Here concavity is interpreted to mean $u''(W) < 0$ for all W. He has initial wealth W_0, and can invest a fraction Y of this in a risky asset yielding a random total rate of return R with a density function $f(R)$ defined over the domain $[R_L, R_H]$. The rest is invested in a safe asset yielding a non-random total rate of return R_0.
   (a) Write down an expression for Apu's random final wealth. (3 points)
   (b) Write down an expression for Apu's expected utility. (3 points)
   (c) Find the first-order condition for Y to maximize expected utility. (3 points)
   (d) Is the second-order condition satisfied? (3 points)
2. **[12 points]** Consider a two-person game in which each player selects a nonnegative number x_i. The payoff function for each player is given by:
$$\pi_i(x_1, x_2) = 2x_1 + 2x_1x_2 - x_i^2 \quad \text{for } i = 1 \text{ and } 2.$$
   (a) Prove that this game has no pure-strategy Nash equilibrium. (3 points)
   (b) Prove that this game has no mixed-strategy Nash equilibrium. (3 points)
   (c) Assume that strategies are restricted to the interval $[0, C]$, where $C > 0$. Find the Nash equilibria of the game. (3 points)
   (d) Assume that strategies are restricted to be the first C positive integers ($C > 1$). Identify the set of strictly dominated strategies and the set of rationalizable strategies. Is the game dominance solvable? (3 points)

Section 3.
Answer one of the following two questions.

1. **[10 points]** Consider a perfectly competitive firm which purchases labor l and machines m, and with these inputs produces an output in the amount of $f(l, m)$. Output sells for a price p. Labor costs w per unit and the price of machines is p_m per unit.

   Unlike the firms discussed in the textbook, this firm is constrained in how much it can spend on labor and on machines. It can only spend at most *CR* dollars on labor and on machines, where *CR*, which is exogenous to the firm, is the amount of credit extended to the firm by its lenders. Its total net costs are the amount it spends on hiring labor, the amount it spends on buying machines, and $r \cdot CR$, where $r > 0$ is the interest rate and hence $r \cdot CR$ is the amount of interest which the firm has to pay back to its lenders. (The loan principal, *CR*, is both part of total revenue and part of total cost, so it cancels out of the expression for profit.)

   (a) Show that the Hessian of the Lagrangian for the profit-maximization problem is

   $$\nabla^2 \mathscr{L} = \begin{bmatrix} 0 & -pf_l' + \lambda w & -pf_m' + \lambda p_m \\ -pf_l' + \lambda w & pf_{ll}'' & pf_{lm}'' \\ -pf_m' + \lambda p_m & pf_{ml}'' & pf_{mm}'' \end{bmatrix} .$$

   Hint: Make use of the first-order conditions.

   (b) Show that if the spending constraint is not binding, the second-order sufficient conditions for a maximum profit for the firm can be expressed in terms of a rather simple property of the production function, and state what the "rather simple property" is.

2. **[10 points]** Suppose a price-taking consumer buys n commodities which are indexed by i, as in: $q_1, q_2, \ldots, q_i, \ldots, q_n$. Let the price of commodity i be p_i. Let the consumer's income be m. Let the consumer's budget share for item i be

   $$\alpha_i = \frac{p_i \, q_i}{m} .$$

   Let

   $$\epsilon_{ij} = \frac{\partial \ln q_i}{\partial \ln p_j} .$$

What is this?

Let

$$\eta_i = \frac{\partial \ln q_i}{\partial \ln m} .$$

What is this?

(a) Prove the so-called Cournot Aggregation Condition:

$$\sum_{j=1}^{n} \alpha_j \, \epsilon_{ji} = -\alpha_i .$$

As a hint: differentiate the budget constraint with respect to p_i.

(b) Prove the so-called Engel Aggregation Condition:

$$\sum_{j=1}^{n} \alpha_j \, \eta_j = 1 .$$

As a hint: differentiate the budget constraint with respect to m.